Graphs

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Today we're going to cover

- Minimum spanning tree
- Shortest paths
- Some known graph problems
- Special graphs
    - Trees
    - Directed acyclic graphs
    - Bipartite graphs

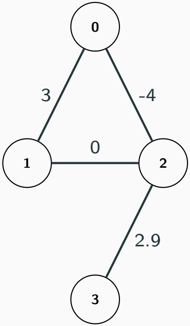
Weighted graphs

- Now the edges in our graphs may have weights, which could represent
  - the distance of the road represented by the edge
  - the cost of going over the edge
  - some capacity of the edge
- We can use a modified adjacency list to represent weighted graphs

Weighted graphs

```
struct edge {
    int u, v;
    int weight;

    edge(int _u, int _v, int _w) {
        u = _u;
        v = _v;
        weight = _w;
    }
};
```



Weighted graphs

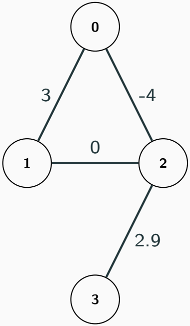
```
vector<edge> adj[4];

adj[0].push_back(edge(0, 1, 3));
adj[0].push_back(edge(0, 2, -4));

adj[1].push_back(edge(1, 0, 3));
adj[1].push_back(edge(1, 2, 0));

adj[2].push_back(edge(2, 0, -4));
adj[2].push_back(edge(2, 1, 0));
adj[2].push_back(edge(2, 3, 2.9));

adj[3].push_back(edge(3, 2, 2.9));
```



Minimum spanning tree

- We have an undirected weighted graph
- The vertices along with a subset of the edges in the graph is called a spanning tree if
    - it forms a tree (i.e. does not contain a cycle) and
    - the tree spans all vertices (all vertices can reach all other vertices)
- The weight of a spanning tree is the sum of the weights of the edges in the subset
- We want to find a minimum spanning tree

Minimum spanning tree

- Several greedy algorithms work

- Go through the edges in the graph in increasing order of weight
- Greedily pick an edge if it doesn't form a cycle (Union-Find can be used to keep track of when we would get a cycle)
- When we've gone through all edges, we have a minimum spanning tree

- This is Kruskal's algorithm
- Time complexity is $O(E \log E)$

Minimum spanning tree

```
bool edge_cmp(const edge &a, const edge &b) {
    return a.weight < b.weight;
}

vector<edge> mst(int n, vector<edge> edges) {
    union_find uf(n);
    sort(edges.begin(), edges.end(), edge_cmp);

    vector<edge> res;
    for (int i = 0; i < edges.size(); i++) {
        int u = edges[i].u,
            v = edges[i].v;

        if (uf.find(u) != uf.find(v)) {
            uf.unite(u, v);
            res.push_back(edges[i]);
        }
    }

    return res;
}
```

Shortest paths

- We have a weighted graph (undirected or directed)
- Given two vertices u, v, what is the shortest path from u to v?

- If all weights are the same, this can be solved with breadth-first search
- Of course, this is usually not the case...

Shortest paths

- There are many known algorithms to find shortest paths
- Like breadth-first search, these algorithms usually find the shortest paths from a given start vertex to all other vertices
- Let's take a quick look at Dijkstra's algorithm, the Bellman-Ford algorithm and the Floyd-Warshall algorithm

Dijkstra's algorithm

```
vector<edge> adj[100];
vector<int> dist(100, INF);

void dijkstra(int start) {
    dist[start] = 0;
    priority_queue<pair<int, int>,
                   vector<pair<int, int> >,
                   greater<pair<int, int> > > pq;
    pq.push(make_pair(dist[start], start));

    while (!pq.empty()) {
        int u = pq.top().second,
            d = pq.top().first;
        pq.pop();
        if (d > dist[u]) continue;
        for (int i = 0; i < adj[u].size(); i++) {
            int v = adj[u][i].v,
                w = adj[u][i].weight;
            if (w + dist[u] < dist[v]) {
                dist[v] = w + dist[u];
                pq.push(make_pair(dist[v], v));
            }
        }
    }
}
```

Dijkstra's algorithm

- Time complexity is $O(V \log E)$
- Note that this only works for non-negative weights

Bellman-Ford algorithm

```
void bellman_ford(int n, int start) {

    dist[start] = 0;

    for (int i = 0; i < n - 1; i++) {
        for (int u = 0; u < n; u++) {
            for (int j = 0; j < adj[u].size(); j++) {
                int v = adj[u][j].v;
                int w = adj[u][j].weight;
                dist[v] = min(dist[v], w + dist[u]);
            }
        }
    }
}
```

Bellman-Ford algorithm

- Time complexity is $O(V \times E)$
- Can be used to detect negative-weight cycles

Floyd-Warshall algorithm

- What about using dynamic programming to compute shortest paths?
- Let $\mathrm{sp}(k, i, j)$ be the shortest path from i to j if we're only allowed to travel through the vertices 0, ..., k
- Base case: $\mathrm{sp}(k, i, j) = 0$ if $i = j$
- Base case: $\mathrm{sp}(-1, i, j) = \mathrm{weight}[a][b]$ if $(i, j) \in E$
- Base case: $\mathrm{sp}(-1, i, j) = \infty$
- $\mathrm{sp}(k, i, j) = \min \begin{cases} \mathrm{sp}(k-1, i, k) + \mathrm{sp}(k-1, k, j) \\ \mathrm{sp}(k-1, i, j) \end{cases}$

Floyd-Warshall algorithm

```c
int dist[1000][1000];
int weight[1000][1000];

void floyd_warshall(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n; j++) {
            dist[i][j] = i == j ? 0 : weight[i][j];
        }
    }

    for (int k = 0; k < n; k++) {
        for (int i = 0; i < n; i++) {
            for (int j = 0; j < n; j++) {
                dist[i][j] = min(dist[i][j], dist[i][k] + dist[k][j]);
            }
        }
    }
}
```

Floyd-Warshall algorithm

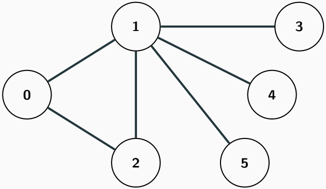
- Computes all-pairs shortest paths
- Time complexity is clearly $O(n^3)$
- Very simple to code

Known graph problems

- The problems we're dealing with very often ask us to solve some well known graph problem
- But usually it's well hidden in the problem statement
- Let's take a look at a few examples

Minimum vertex cover

- We have an unweighted undirected graph
- A vertex cover is a subset of the vertices S, such that for each edge (u, v) in the graph, either u or v (or both) are in S



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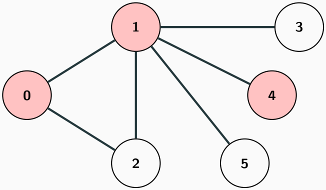

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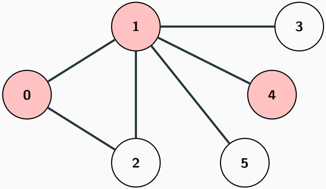

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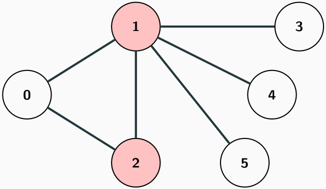

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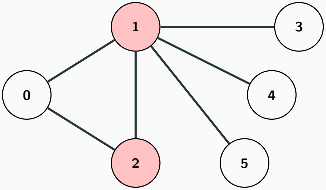

- We want to find a vertex cover of minimum size
- NP-hard problem in general graphs

Maximum independent set

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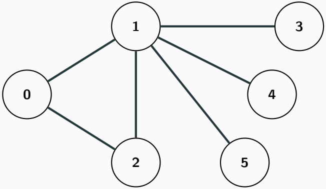
- An independent set is a subset of the vertices S, such that no two vertices u, v in S are adjacent in the graph



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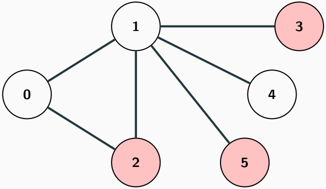

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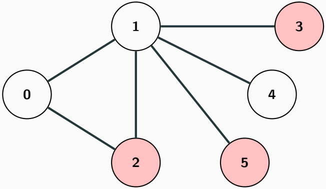

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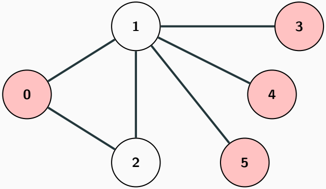

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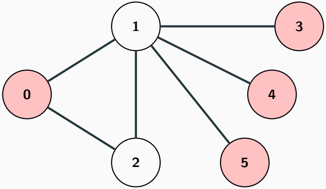
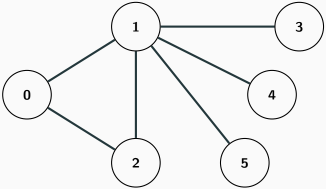


- We want to find an independent set of maximum size
- NP-hard problem in general graphs

Relation between MVC and MIS

- The previous two problems are very related
- A subset of the vertices is a vertex cover if and only if the complement of the set is an independent set



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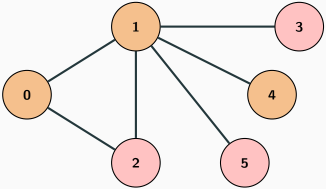

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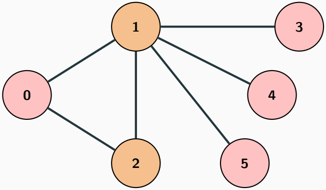

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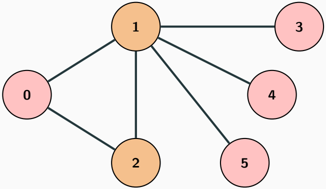
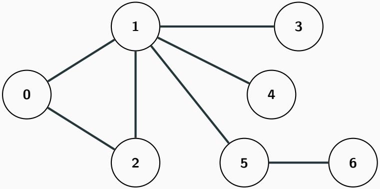


- The size of a minimum vertex cover plus the size of a maximum independent set is equal to the number of vertices

Maximum matching

- We have an unweighted undirected graph
- A matching is a subset of the edges such that each vertex is adjacent to at most one edge in the subset



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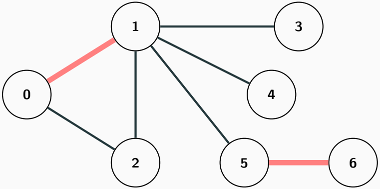

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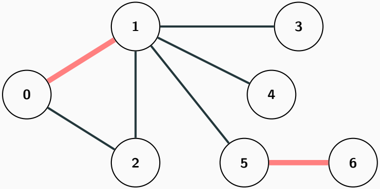

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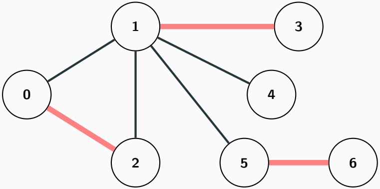

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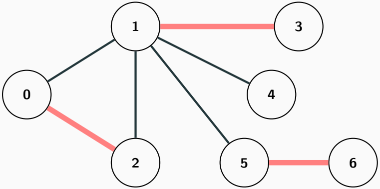
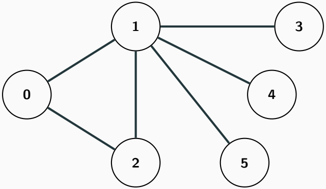


- We want to find a matching of maximum size
- There exists an $O(V^4)$ algorithm for general graphs, but is pretty complex

Graph coloring

- We have an unweighted undirected graph
- A coloring of the graph is an assignment of colors to the vertices such that adjacent vertices have different colors



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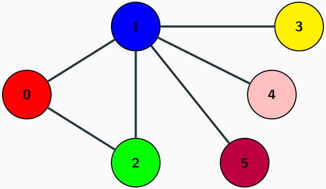

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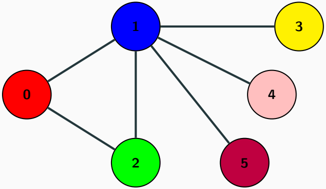

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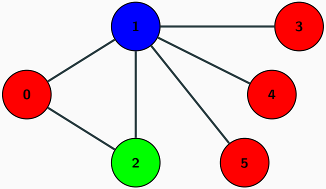

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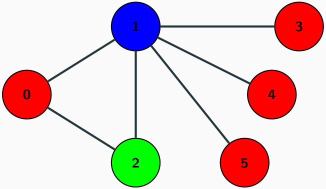


- We want to find a coloring that uses the minimum number of distinct colors

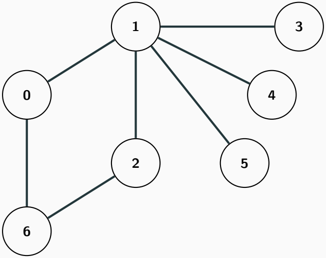
- NP-hard in general graphs

Special graphs

- All of these problems are hard (in some sense) in general graphs
- But what if we're working with special kinds of graphs?
- Let's look at a few examples

Bipartite graphs

- A graph is bipartite if the vertices can be partitioned into two sets such that for each edge (u, v) u and v are in different sets



- How do we check if a graph is bipartite?

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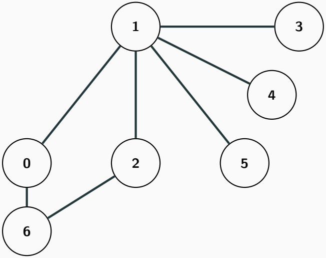

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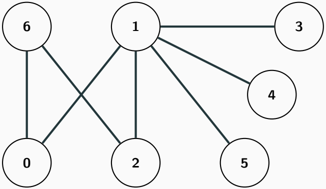

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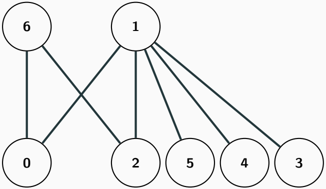

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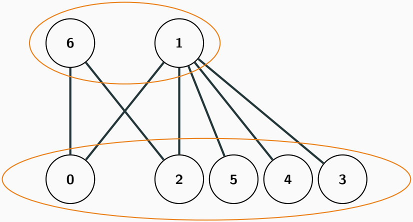


- How do we check if a graph is bipartite?

Bipartite graphs

- We want to check if we can split the vertices into these two groups
- Take any vertex, and assume that it's in the first group
- Then all of his neighbors must be in the second group
- And then all of their neighbors must be in the first group
- And so on...

- We can do this with a simple depth-first search
- If we ever find a contradiction (i.e. a vertex must both be in the first and second set), then the graph is not bipartite

Bipartite graphs

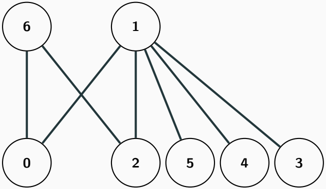
```
vector<int> adj[1000];
vector<int> side(1000, -1);
bool is_bipartite = true;

void check_bipartite(int u) {
    for (int i = 0; i < adj[u].size(); i++) {
        int v = adj[u][i];
        if (side[v] == -1) {
            side[v] = 1 - side[u];
            check_bipartite(v);
        } else if (side[u] == side[v]) {
            is_bipartite = false;
        }
    }
}

for (int u = 0; u < n; u++) {
    if (side[u] == -1) {
        side[u] = 0;
        check_bipartite(u);
    }
}
```

Coloring bipartite graphs

- What if we want to find the minimum graph coloring of a bipartite graph?



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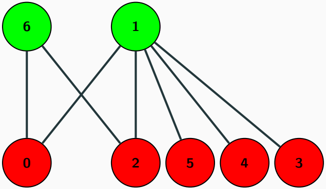

- Simple, one side can be colored with one color, and the second side can be colored with a second color

Bipartite matching

- Finding a maximum matching in bipartite graphs is very common
- *see example*

- Soon we'll see an efficient algorithm for finding the maximum matching in a bipartite graph

König's theorem

- König's theorem states that the size of a minimum vertex cover in a bipartite graph is equal to the size of the maximum matching in that graph
- To find the minimum vertex cover in a bipartite graph, we just find the maximum matching with our efficient algorithm, and we have our answer
- And since the size of the maximum independent set is just the number of vertices minus the size of the minimum vertex cover, we can also compute the maximum independent set for a bipartite graph efficiently

Trees

- An undirected graph is a tree if it has no cycles
- Easy to check if a graph is a tree by checking if there are any backward edges in the depth-first search tree (see previous lecture)
- A connected tree with n vertices has exactly $n - 1$ edges
- Between each pair of vertices u, v in the tree, there exists exactly one simple path, which can be found with depth-first search (or breadth-first search)

Trees

- What if we look at these problems for trees?
- How do we find the minimum number of colors needed to color a tree?

Trees

- What if we look at these problems for trees?
- How do we find the minimum number of colors needed to color a tree?
- Well, trees are actually bipartite graphs...
- Why? Pick some vertex and make it the root of the tree. Then vertices at even heights in the tree can be put on one side, and vertices at odd heights can be put on the other side
- So all the efficient algorithms for bipartite graphs also work for trees

Trees

- Trees are also well suited for dynamic programming, so many problems become simpler here because of that

Directed acyclic graphs

- A directed graph is a directed acyclic graph if it doesn't contain any cycles
- Easy to check if a graph is a DAG by checking if there are any backward edges in the depth-first search tree (see previous lecture)
- Many problems are simple on DAGs, since it's easy to do dynamic programming over DAGs
  - counting number of simple paths from u to v
  - longest simple path from u to v